Failure rate estimation for semi-Markov chains

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Abstract

We consider a semi-Markov chain, with a finite state space. Taking a censored history, we obtain empirical estimators for the discrete semi-Markov kernel, renewal function and semi-Markov transition function. We propose estimators for two different failure rate functions. We study the strong consistency and the asymptotic normality for each estimator and we construct the confidence intervals. We illustrate our results by a numerical example.

1 Preliminaries

Consider a finite state space $E = \{1, 2, \ldots, s\}, s < \infty$, and $(J_n, S_n)_{n\in\mathbb{N}}$ a homogeneous discrete time Markov renewal process ($DTMRP$), where $(J_n)_{n\in\mathbb{N}}$, the visited states of the system, is an $E-$ valued Markov chain and $(S_n)_{n\in\mathbb{N}}$, the jump times of the process, is an $\mathbb{N}-$ valued process. Put $X_n := S_n - S_{n-1}, n \in \mathbb{N}^*$, for the inter-jump times. The $DTMRP$ is defined by the initial distribution $\alpha(i) := \mathbb{P}(J_0 = i)$ and by the semi-Markov kernel $q = (q_{ij}(\cdot))_{i,j\in E}$, $q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} = k \mid J_n = i),\ k \in \mathbb{N}$, where $q_{ij}(0) := 0$.

Denote by $V = (p_{ij})_{i,j\in E}$, $p_{ij} := \mathbb{P}(J_{n+1} = j \mid J_n = i) = \sum_{k\geq 0} q_{ij}(k)$, the transition matrix of $(J_n)_{n\in\mathbb{N}}$ and by $f = (f_{ij}(\cdot))_{i,j\in E}$ the conditional distribution matrix of the sojourn times, $f_{ij}(k) := \mathbb{P}(X_{n+1} = k \mid J_n = i, J_{n+1} = j) = q_{ij}(k)/p_{ij}$, for $p_{ij} \neq 0$. Consider the matrix-valued functions $Q = (Q_{ij}(\cdot))_{i,j\in E}$ and $\psi = (\psi_{ij}(\cdot))_{i,j\in E}$ defined by $Q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} \leq k \mid J_n = i) = \sum_{l=0}^{k} q_{ij}(l)$ and by $\psi_{ij}(k) := \sum_{n=0}^{k} q_{ij}^{(n)}(k),\ k \in \mathbb{N}$, where all the powers are in the matrix convolution sense. Let also $\Psi = (\Psi_{ij}(\cdot))_{i,j\in E}$ be the matrix renewal function defined by $\Psi_{ij}(k) := \sum_{n=0}^{k} Q_{ij}^{(n)}(k) = \sum_{l=0}^{k} \psi_{ij}(l)$.

The semi-Markov chain $(Z_k)_{k\in\mathbb{N}}$ associated to the DTMRP (J, S) is defined by $Z_k := J_{N(k)},\ k \in \mathbb{N}$, where $N(k) := \max\{n \geq 0 \mid S_n \leq k\}$ is the discrete time counting process of the number of jumps in $[1, k] \subset \mathbb{N}$. Let $P = \big(P_{ij}(\cdot)\big)_{i,j\in E}$ be the transition matrix of the semi-Markov process Z, $P_{ij}(k) := \mathbb{P}(Z_k = j \mid Z_0 = i),\ k \in \mathbb{N}$. The associated Markov renewal equation is (see, e.g., Barbu et al. 2004)

$$P(k) = I - diag(Q(k) \cdot \mathbf{1}_s) + q * P(k), k \in \mathbb{N}, \tag{1}$$

where $\mathbf{1}_s := (1, \ldots, 1)^t$ and $diag(Q(k)\cdot\mathbf{1}_s)$ is the diagonal matrix having the vector $Q(k)\cdot\mathbf{1}_s$ as main diagonal. Solving the Markov renewal equation (1) we obtain that the unique solution is

$$P(k) = \Big[\psi * (I - diag(Q\ \cdot \mathbf{1}_s))\Big](k). \tag{2}$$

All the results are given for a DTMRP which satisfies the following assumptions:

A1. The Markov chain $(J_n)_{n\in\mathbb{N}}$ is irreducible;

A2. The mean sojourn times are finite, i.e., $\sum_{k\geq 0} k h_i(k) < \infty$ for any state $i \in E$, where $h_i(k) := \mathbb{P}(X_{n+1} = k \mid J_n = i)$ is the sojourn time distribution in state i;

A3. The $DTMRP$ $(J_n, S_n)_{n\in\mathbb{N}}$ is aperiodic.

Let us consider a sample path $\boldsymbol{H}(M) := (J_0, X_1, \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, U_M)$ of the DTMRP $(J_n, S_n)_{n\in\mathbb{N}}$, censored at time $M \in \mathbb{N}$, where we set $U_M := M - S_{N(M)}$.

For all $i,j \in E$ and $k \in \mathbb{N}, k \leq M$, we define the empirical estimator of the discrete time semi-Markov kernel $q_{ij}(k)$ by $\hat{q}_{ij}(k,M) := N_{ij}(k,M)/N_i(M)$, where $N_i(M) := \sum_{n=0}^{N(M)-1} \mathbf{1}_{\{J_n=i\}}$ is the number of visits to state i, up to time M and $N_{ij}(k,M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n=k\}}$ is the number of transitions from i to j, up to time M, with sojourn time in state i equal to k. We stress the fact that the above empirical estimator is an approached nonparametric maximum likelihood estimator, i.e., it maximizes the approached likelihood function, obtained by neglecting the part corresponding to $U_M = M - S_{N(M)}$.

Replacing q by its estimator in the expressions of Q, ψ, Ψ and P we obtain their corresponding estimators, noted by $\hat{Q}, \hat{\psi}, \hat{\Psi}, \hat{P}$. We can prove the strong consistency for the proposed estimators and the asymptotic normality of $\hat{q}_{ij}(k,M)$, $\hat{\psi}_{ij}(k,M)$, $\hat{P}_{ij}(k,M)$ (see Barbu and Limnios 2004b).

2 Failure rates estimation

Denote by T the lifetime of the system. Let E be partitioned into two subsets, $U = \{1,\dots,s_1\}$ for the up states and $D = \{s_1+1,\dots,s\}, 0 < s_1 < s$, for the down states, with $E = U \cup D$ and $U \cap D = \emptyset$. We will write all vectors and matrix-valued functions according to this partition. For instance, we have

$$P(k) = \begin{array}{cc} \begin{array}{cc} \quad U & \qquad D \end{array} & \\ \begin{pmatrix} P_{11}(k) & P_{12}(k) \\ P_{21}(k) & P_{22}(k) \end{pmatrix} & \begin{array}{c} U \\ D \end{array} \end{array} \qquad \alpha = \begin{array}{cc} U & D \\ (\alpha_1 & \alpha_2) \end{array}.$$

The reliability of the system at the moment $k \in \mathbb{N}$ is given by (see Barbu et al. 2004a)

$$R(k) \;=\; \alpha_1 \cdot P_{11}(k) \cdot \mathbf{1}_{s_1} = \alpha_1\, \psi_{11} * (I - diag(Q \cdot \mathbf{1}_s)_{11}) \mathbf{1}_{s_1}.$$

We obtain the following estimator for the system reliability:

$$\hat{R}(k,M) \;:=\; \alpha_1 \cdot \hat{P}_{11}(k,M) \cdot \mathbf{1}_{s_1} = \alpha_1 \Big[\hat{\psi}_{11}(\cdot,M) * \Big(I - diag(\hat{Q}(\cdot,M) \cdot \mathbf{1}_s)_{11}\Big)\Big](k)\mathbf{1}_{s_1}. \tag{3}$$

In the sequel, we consider two different definitions of the failure rate function.

• $BMP-$**failure rate function** $\lambda(k)$

It is the usual failure rate, defined by Barlow et al. (1963) as the conditional probability that the failure of the system occurs at time k, given that the system has worked until time $k-1$ (for the $BMP-$failure rate of Markov chains, see Sadek and Limnios 2002; for the $BMP-$failure rate of continuous semi-Markov systems see Ouhbi and Limnios 1998). The $BMP-$ failure rate is defined by

$$\lambda(k) \;:=\; \mathbb{P}(T = k \mid T \geq k) = \begin{cases} 1 - R(k)/R(k-1), & R(k-1) \neq 0 \\ 0, & otherwise \end{cases}, k \geq 1, \quad \text{and} \quad \lambda(0) := 1 - R(0).$$

• $RG-$**failure rate function** $r(k)$

A new definition of the discrete failure rate is proposed by Roy and Gupta (1992) for solving some of the problems raised by the use of the usual failure rate function $\lambda(k)$ in discrete time. A detailed argument for the introduction of the new definition of the failure rate function is given in Bracquemond et al. (2001) (see Sadek and Limnios 2002 for the RG-failure rate of Markov chains). The $RG-$ failure rate is defined by

$$r(k) := \begin{cases} \ln\big[R(k-1)/R(k)\big], & k \geq 1 \\ -\ln R(0), & \text{k} = 0 \end{cases}.$$

The two failure rates are related by $r(k) = -\ln(1 - \lambda(k))$. We propose the following estimators:

$$\hat{\lambda}(k,M) \;:=\; \begin{cases} 1 - \frac{\hat{R}(k,M)}{\hat{R}(k-1,M)}, & \hat{R}(k-1,M) \neq 0 \\ 0, & otherwise \end{cases}, k \geq 1, \quad \hat{r}(k,M) := \begin{cases} \ln \frac{\hat{R}(k-1,M)}{\hat{R}(k,M)}, & k \geq 1 \\ -\ln \hat{R}(0,M), & \text{k} = 0 \end{cases}.$$

$$\hat{\lambda}(0,M) \;:=\; 1 - \hat{R}(0,M),$$

Theorem 1 *For any fixed $k \in \mathbb{N}$, the BMP$-$failure rate estimator $\hat{\lambda}(k,M)$ is strongly consistent, as M tends to infinity, and we have $\sqrt{M}[\hat{\lambda}(k,M)-\lambda(k)] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma_\lambda^2(k))$, where*

$$
\begin{aligned}
\sigma_\lambda^2(k) &= \sigma_1^2(k)/R^4(k-1),\\
\sigma_1^2(k) &= \frac{\mu_{jj}^*}{\mu_{jj}}\sum_{i=1}^{s}\frac{\mu_{ii}^2}{\mu_{ii}^*}\Big\{R^2(k)\sum_{j=1}^{s}\Big[D_{ij}^U-\mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\Psi_{ti}\Big]^2 * q_{ij}(k-1)\\
&\quad +R^2(k-1)\sum_{j=1}^{s}\Big[D_{ij}^U-\mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\Psi_{ti}\Big]^2 * q_{ij}(k)-T_i^2(k)\\
&\quad +2R(k-1)R(k)\sum_{j=1}^{s}\Big[\mathbf{1}_{\{i\in U\}}D_{ij}^U\sum_{t\in U}\alpha(t)\Psi_{ti}^+ +\mathbf{1}_{\{i\in U\}}(D_{ij}^U)^+\sum_{t\in U}\alpha(t)\Psi_{ti}\\
&\quad -(D_{ij}^U)^+D_{ij}^U-\mathbf{1}_{\{i\in U\}}\Big(\sum_{t\in U}\alpha(t)\Psi_{ti}\Big)\Big(\sum_{t\in U}\alpha(t)\Psi_{ti}^+\Big)\Big] * q_{ij}(k-1)\Big\},
\end{aligned}
\tag{4}
$$

with

$$
\begin{aligned}
T_i(k) &:= \sum_{j=1}^{s}\Big[R(k)D_{ij}^U * q_{ij}(k-1)-R(k-1)D_{ij}^U * q_{ij}(k)\\
&\quad -R(k)\mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\psi_{ti} * Q_{ij}(k-1)+R(k-1)\mathbf{1}_{\{i\in U\}}\sum_{t\in U}\alpha(t)\psi_{ti} * Q_{ij}(k)\Big],\\
D_{ij}^U &:= \sum_{n\in U}\sum_{r\in U}\alpha(n)\psi_{ni} * \psi_{jr} * \Big(I-diag(Q\cdot\mathbf{1})\Big)_{rr},
\end{aligned}
$$

μ_{ii}^* is the mean recurrence time of state i for the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$, μ_{ii} is the mean recurrence time of the state i for the DTMRP (J,S) and $A^+(k) := A(k+1)$ for $A = \big(A_{ij}(\cdot)\big)_{i,j\in E}$.

Corollary 1 *For any fixed $k \in \mathbb{N}$, the RG$-$failure rate estimator $\hat{r}(k,M)$ is strongly consistent, as M tends to infinity, and we have $\sqrt{M}[\hat{r}(k,M)-r(k)] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma_r^2(k))$, where*

$$\sigma_r^2(k) = \sigma_1^2(k)/\big[R^2(k-1)R^2(k)\big], \quad \textit{with } \sigma_1^2(k) \textit{ given in Equation (4).}$$

Replacing $q(k), Q(k), \psi(k), \Psi(k)$ by their estimators in Equation (4), we obtain a strongly consistent estimator $\hat{\sigma}_\lambda^2(k)$ of the variance $\sigma_\lambda^2(k)$. Let $\gamma \in (0,1)$ and u_γ be the $\gamma-$ quantile of $N(0,1)$. For $k \in \mathbb{N}, k \leq M$, the estimated asymptotic confidence interval of BMP-failure rate $\lambda(k)$ at level $100(1-\gamma)\%$ is given by

$$\hat{\lambda}(k,M)-u_{1-\gamma/2}\hat{\sigma}_\lambda(k)/\sqrt{M} \leq \lambda(k) \leq \hat{\lambda}(k,M)+u_{1-\gamma/2}\hat{\sigma}_\lambda(k)/\sqrt{M}.$$

3 Numerical example

Let the state space $E = \{1,2,3\}$ be partitioned into the up-state set $U = \{1,2\}$ and the down-state set $D = \{3\}$. The system is defined by the initial distribution $\mu := (1\ 0\ 0)$, by the transition probability matrix V of $(J_n)_{n\in\mathbb{N}}$ and by the conditional distribution matrix of the sojourn times f given by

$$V := \begin{pmatrix} 0 & 1 & 0\\ 0.95 & 0 & 0.05\\ 1 & 0 & 0 \end{pmatrix}, \quad f(k) := \begin{pmatrix} 0 & f_{12}(k) & 0\\ f_{21}(k) & 0 & f_{23}(k)\\ f_{31}(k) & 0 & 0 \end{pmatrix},\ k \in \mathbb{N},$$

- f_{12} is a geometric distribution defined by $f_{12}(0) := 0$, $f_{12}(k) := p(1-p)^{k-1}, k \geq 1$, with $p = 0.8$,

- $f_{21} := W_{q_1,b_1}, f_{23} := W_{q_2,b_2}$ and $f_{31} := W_{q_3,b_3}$ are discrete time, first type, Weibull distributions, defined by $W_{q,b}(0) := 0$, $W_{q,b}(k) := q^{(k-1)^b} - q^{k^b}, k \geq 1$, with $q_1 = 0.5, b_1 = 0.7, q_2 = 0.6, b_2 = 0.9, q_3 = 0.5, b_3 = 2$ (for discrete time Weibull distribution, see, e.g., Nakagawa and Osaki 1975).

Figure 1 gives the empirical estimator and the confidence interval at level 95% for the $BMP-$failure rate (left) and a comparison between $BMP-$failure rate estimators obtained for different sample sizes (right).

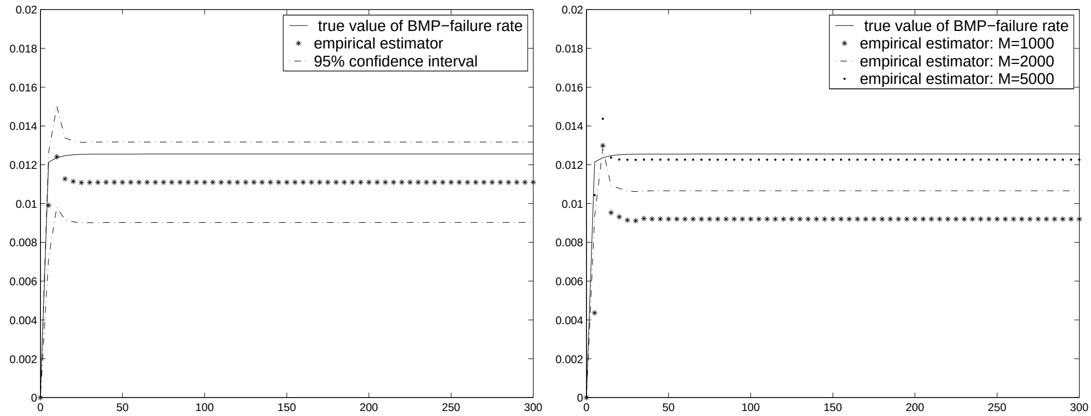

Figure 1: $BMP-$ failure rate confidence interval and $BMP-$ failure rate estimator consistency

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